

Compact Inverses of The Multipoint Normal Differential Operators For First Order

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Abstract

In this work, firstly all normal extensions of a multipoint minimal operator generated by linear multipoint differential-operator expression for first order in the Hilbert space of vector functions in terms of boundary values at the endpoints of the infinitely many separated subintervals are described. Finally, a compactness properties of the inverses of such extensions has been investigated.

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1 Introduction

It is known that traditional infinite direct sum of Hilbert spaces H_n , $n \geq 1$ and infinite direct sum of operators A_n in H_n , $n \geq 1$ are defined as

$$H = \bigoplus_{n=1}^{\infty} H_n = \left\{ u = (u_n) : u_n \in H_n, n \geq 1 \text{ and } \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < +\infty \right\},$$

$$A = \bigoplus_{n=1}^{\infty} A_n, D(A) = \{ u = (u_n) \in H : u_n \in D(A_n), n \geq 1 \text{ and } Au = (A_n u_n) \in H \}.$$

A linear space H is a Hilbert space with norm induced by the inner product

$$(u, v)_H = \sum_{n=1}^{\infty} (u_n, v_n)_{H_n}, \quad u, v \in H \quad [1].$$

The general theory of linear closed operators in Hilbert spaces and its applications to physical problems have been investigated by many researches (for example, see [1], [2]).

However, many physical problems requires the study of the theory of linear operators in direct sums in Hilbert spaces (for example, see [3]-[8] and references therein).

We note that a detail analysis of normal subspaces and operators in Hilbert spaces have been

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studied in [9] (see references in it).

This study contains three section except introduction. In section 2, the multipoint minimal and maximal operators for the first order differential-operator expression are determined. In section 3, all normal extensions of multipoint formally normal operators are described in terms of boundary values in the endpoints of the infinitely many separated subintervals. Finally In section 4, compactness properties of the inverses of such extensions have been established.

2 The Minimal and Maximal Operators

Throughout this work (a_n) and (b_n) will be sequences of real numbers such that

$$-\infty < a_n < b_n < a_{n+1} < \cdots < +\infty,$$

H_n is any Hilbert space, $\Delta_n = (a_n, b_n)$, $L_n^2 = L^2(H_n, \Delta_n)$, $L^2 = \bigoplus_{n=1}^{\infty} L^2(H_n, \Delta_n)$, $n \geq 1$, $\sup_{n \geq 1} (b_n - a_n) < +\infty$, $W_2^1 = \bigoplus_{n=1}^{\infty} W_2^1(H_n, \Delta_n)$, $W_2^0 = \bigoplus_{n=1}^{\infty} W_2^0(H_n, \Delta_n)$, $H = \bigoplus_{n=1}^{\infty} H_n$, $cl(T)$ -closure of the operator T . $l(\cdot)$ is a linear multipoint differential-operator expression for first order in L^2 in the following form

$$l(u) = (l_n(u_n)) \quad (2.1)$$

and for each $n \geq 1$

$$l_n(u_n) = u_n' + A_n u_n, \quad (2.2)$$

where $A_n : D(A_n) \subset H_n \rightarrow H_n$ is a linear positive defined selfadjoint operator in H_n .

It is clear that formally adjoint expression to (2.2) in the Hilbert space L_n^2 is in the form

$$l_n^+(v_n) = -v_n' + A_n v_n, \quad n \geq 1. \quad (2.3)$$

We define an operator L_{n0}' on the dense manifold of vector functions D_{n0}' in L_n^2 as

$$D_{n0}' := \left\{ u_n \in L_n^2 : u_n = \sum_{k=1}^m \phi_k f_k, \phi_k \in C_0^\infty(\Delta_n), f_k \in D(A_n), k = 1, 2, \dots, m; m \in \mathbb{N} \right\}$$

with $L_{n0}' u_n := l_n(u_n)$, $n \geq 1$.

Since the operator $A_n > 0$, $n \geq 1$, then from the relation

$$Re(L_{n0}' u_n, u_n)_{L_n^2} = 2(A_n u_n, u_n)_{L_n^2} \geq 0, \quad u_n \in D_{n0}'$$

it implies that L_{n0}' is an accretive in L_n^2 , $n \geq 1$. Hence the operator L_{n0}' has a closure in L_n^2 , $n \geq 1$. The closure $cl(L_{n0}')$ of the operator L_{n0}' is called the minimal operator generated by differential-

operator expression (2.2) and is denoted by L_{n0} in L_n^2 , $n \geq 1$. The operator L_0 is defined by

$$D(L_0) := \left\{ u = (u_n) : u_n \in D(L_{n0}), n \geq 1, \sum_{n=1}^{\infty} \|L_{n0}u_n\|_{L_n^2}^2 < +\infty \right\}$$

with

$$L_0 u := (L_{n0}u_n), u \in D(L_0), L_0 : D(L_0) \subset L^2 \rightarrow L^2$$

is called a minimal operator (multipoint) generated by differential-operator expression (2.1) in Hilbert space L^2 and denoted by $L_0 = \bigoplus_{n=1}^{\infty} L_{n0}$.

In a similar way the minimal operator for twopoints denoted by L_{n0}^+ in L_n^2 , $n \geq 1$ for the formally adjoint linear differential-operator expression (2.3) can be constructed.

In this case the operator L_0^+ defined by

$$D(L_0^+) := \left\{ v := (v_n) : v_n \in D(L_{n0}^+), n \geq 1, \sum_{n=1}^{\infty} \|L_{n0}^+v_n\|_{L_n^2}^2 < +\infty \right\}$$

with $L_0^+ v := (L_{n0}^+v_n)$, $v \in D(L_0^+)$, $L_0^+ : D(L_0^+) \subset L^2 \rightarrow L^2$ is called a minimal operator (multipoint) generated by $l^+(v) = (l_n^+(v_n))$ in the Hilbert space L^2 and denoted by $L_0^+ = \bigoplus_{n=1}^{\infty} L_{n0}^+$.

We now state the following relevant result.

Theorem 2.1. The minimal operators L_0 and L_0^+ are densely defined closed operators in L^2 .

The following defined operators in L^2 $L := (L_0^+)^* = \bigoplus_{n=1}^{\infty} L_n$ and $L^+ := (L_0)^* = \bigoplus_{n=1}^{\infty} L_n^+$ are called maximal operators (multipoint) for the differential-operator expression $l(\cdot)$ and $l^+(\cdot)$ respectively. It is clear that $Lu = (l_n(u_n))$, $u \in D(L)$,

$$D(L) := \left\{ u = (u_n) \in L^2 : u_n \in D(L_n), n \geq 1, \sum_{n=1}^{\infty} \|L_n u_n\|_{L_n^2}^2 < \infty \right\},$$

$$L^+ v = (l_n^+(v_n)), v \in D(L^+),$$

$$D(L^+) := \left\{ v = (v_n) \in L^2 : v_n \in D(L_n^+), n \geq 1, \sum_{n=1}^{\infty} \|L_n^+ v_n\|_{L_n^2}^2 < \infty \right\}$$

and $L_0 \subset L$, $L_0^+ \subset L^+$.

Furthermore, the validity of following proposition is clear.

Theorem 2.2. The domain of the operators L and L_0 are

$$\begin{aligned} D(L) = \{ & u = (u_n) \in L^2 : (1) \text{ for each } n \geq 1 \text{ vector function } u_n \in L_n^2, u_n \\ & \text{is absolutely continuous in interval } \Delta_n; \\ & (2) l_n(u_n) \in L_n^2, n \geq 1; (3) l(u) = (l_n(u_n)) \in L^2 \} \\ = \{ & u = (u_n) \in L^2 : u_n \in D(L_n), n \geq 1 \text{ and } l(u) = (l_n(u_n)) \in L^2 \}, \end{aligned}$$

$$D(L_0) = \{ u = (u_n) \in D(L) : u_n(a_n) = u_n(b_n) = 0, n \geq 1 \}.$$

Remark 2.3. If $A_n \in B(H)$, $n \geq 1$ and $\sup_{n \geq 1} \|A_n\| \leq c < +\infty$, then for any $u = (u_n) \in L^2$ we have $(Au) = (A_n u_n) \in L^2$.

Now the following results can be proved .

Theorem 2.4. If a minimal operator L_0 is formally normal in L^2 , then $D(L_0) \subset \overset{0}{W}_2^1$ and $AD(L_0) \subset L^2$.

Theorem 2.5. If $A^{1/2}W_2^1 \subset W_2^1$, then minimal operator L_0 is formally normal in L^2 .

Proof: In this case from the following relations

$$L_0^+ u = L_0 u - 2Au, \quad u \in D(L_0),$$

$$L_0 u = L_0^+ u + 2Au, \quad u \in D(L_0^+)$$

it implies that $D(L_0) = D(L_0^+)$. Since $D(L_0^+) \subset D(L_0^*) = D(L^+)$, it is obtained that $D(L_0) \subset D(L^+)$.

On the other hand for any $u \in D(L_0)$

$$\begin{aligned} \|L_0 u\|_{L^2}^2 &= (u' + Au, u' + Au)_{L^2} = \|u'\|_{L^2}^2 + [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|_{L^2}^2 \\ &= \|u'\|_{L^2}^2 + \|Au\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} \|L^+ u\|_{L^2}^2 &= (-u' + Au, -u' + Au)_{L^2} = \|u'\|_{L^2}^2 - [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|_{L^2}^2 \\ &= \|u'\|_{L^2}^2 + \|Au\|_{L^2}^2. \end{aligned}$$

Thus, it is established that operator L_0 is formally normal in L^2 .

Remark 2.6. If $A_n \in B(H)$, $n \geq 1$ and $\sup_{n \geq 1} \|A_n\| \leq c < +\infty$, then $D(L_0) = D(L_0^+)$ and $D(L) = D(L^+)$.

Remark 2.7. If $AW_2^1 \subset L^2$, then $D(L_0) = D(L_0^+)$ and $D(L) = D(L^+)$.

3 Description of Normal Extensions of the Minimal Operator

In this section the main purpose is to describe all normal extensions of the minimal operator L_0 in L^2 in terms in the boundary values of the endpoints of the subintervals .

First, we will show that there exists normal extension of the minimal operator L_0 . Consider the following extension of the minimal operator L_0

$$\left\{ \begin{array}{l} \tilde{L}u := u' + Au, \quad AW_2^1 \subset W_2^1, \\ D(\tilde{L}) = \{u = (u_n) \in W_2^1 : u_n(a_n) = u_n(b_n), \quad n \geq 1\}. \end{array} \right.$$

Under the condition on the coefficient A we have

$$\begin{aligned} (\tilde{L}u, v)_{L^2} &= (u', v)_{L^2} + (Au, v)_{L^2} = (u, v)'_{L^2} + (u, -v' + Av)_{L^2} \\ &= \sum_{n=1}^{\infty} [(u_n(b_n), v_n(b_n))_{H_n} - (u_n(a_n), v_n(a_n))_{H_n}] + (u, -v' + Av)_{L^2} \end{aligned}$$

From this it is obtained

$$\begin{cases} \tilde{L}^*v := -v' + Av, \\ D(\tilde{L}^*) = \{v = (v_n) \in W_2^1 : v_n(a_n) = v_n(b_n), n \geq 1\}. \end{cases}$$

In this case it is clear that $D(\tilde{L}) = D(\tilde{L}^*)$. On the other hand, since for each $u \in D(\tilde{L})$

$$\begin{aligned} \|\tilde{L}u\|_{L^2}^2 &= \|u'\|_{L^2}^2 + [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|_{L^2}^2, \\ \|\tilde{L}^*u\|_{L^2}^2 &= \|u'\|_{L^2}^2 - [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} (u', Au)_{L^2} + (Au, u')_{L^2} &= (u, Au)'_{L^2} \\ &= \sum_{n=1}^{\infty} [(u_n(b_n), A_n u_n(b_n))_{H_n} - (u_n(a_n), A_n u_n(a_n))_{H_n}] = 0. \end{aligned}$$

Then $\|\tilde{L}u\|_{L^2} = \|\tilde{L}^*u\|_{L^2}$ for every $u \in D(\tilde{L})$. Consequently, \tilde{L} is a normal extension of the minimal operator L_0 .

The following result establishes the relationship between normal extensions of L_0 and normal extensions of L_{n0} , $n \geq 1$.

Theorem 3.1. The extension $\tilde{L} = \bigoplus_{n=1}^{\infty} \tilde{L}_n$ of the minimal operator L_0 in L^2 is a normal if and only if for any $n \geq 1$, \tilde{L}_n is so in L_n^2 .

Now using the Theorem 3.1 and [10] we can formulate the following main result of this section, where it is given a description of all normal extension of the minimal operator L_0 in L^2 in terms of boundary values of vector functions at the endpoints of subintervals.

Theorem 3.2. Let $A^{1/2}W_2^1 \subset W_2^1$. If $\tilde{L} = \bigoplus_{n=1}^{\infty} \tilde{L}_n$ is a normal extension of the minimal operator L_0 in L^2 , then it is generated by differential-operator expression (2.1) with boundary conditions

$$u_n(b_n) = W_n u_n(a_n), u_n \in D(L_n), \quad (3.1)$$

where W_n is a unitary operator in H_n and $W_n A_n^{-1} = A_n^{-1} W_n, n \geq 1$. The unitary operator $W = \bigoplus_{n=1}^{\infty} W_n$ in $H = \bigoplus_{n=1}^{\infty} H_n$ is determined uniquely by the extension \tilde{L} , that is $\tilde{L} = L_W$.

On the contrary, the restriction of the maximal operator L to the linear manifold $u \in D(L)$ satisfying the condition (3.1) with any unitary operator $W = \bigoplus_{n=1}^{\infty} W_n$ in H with property $W A^{-1} = A^{-1} W$ is a normal extension of the minimal operator L_0 in L^2 .

4 Some Compactness Properties of The Normal Extensions

The following two proposition can be easily proved in general case.

Theorem 4.1. For the point spectrum of $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$ in the direct sum $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ of Hilbert spaces $\mathcal{H}_n, n \geq 1$ it is true that

$$\sigma_p(\mathcal{A}) = \bigcup_{n=1}^{\infty} \sigma_p(\mathcal{A}_n)$$

Theorem 4.2. Let $\mathcal{A}_n \in B(\mathcal{H}_n), n \geq 1, \mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$ and $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$. In order for $\mathcal{A} \in B(\mathcal{H})$ the necessary and sufficient condition is that the $\sup_{n \geq 1} \|\mathcal{A}_n\|$ be finite. In this case $\|\mathcal{A}\| = \sup_{n \geq 1} \|\mathcal{A}_n\|$.

Let $C_{\infty}(\cdot)$ and $C_p(\cdot), 1 \leq p < \infty$ denote the class of compact operators and the Schatten-von Neumann subclasses of compact operators in corresponding spaces respectively.

Definition 4.3.[11] Let T be a linear closed and densely defined operator in any Hilbert space \mathfrak{H} . If $\rho(T) \neq \emptyset$ and for $\lambda \in \rho(T)$ the resolvent operator $R_{\lambda}(T) \in C_{\infty}(\mathfrak{H})$, then operator $T : D(T) \subset \mathfrak{H} \rightarrow \mathfrak{H}$ is called an operator with discrete spectrum

In first note that the following results are true.

Theorem 4.4. If the operator $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$ as an operator with discrete spectrum in $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, then for every $n \geq 1$ the operator \mathcal{A}_n is so in \mathcal{H}_n .

Remark 4.5. Unfortunately, the converse of the Theorem 4.4 is not true in general case.

Indeed, consider the following sequence of operators $\mathcal{A}_n u_n = u_n, 0 < \dim \mathcal{H}_n = d_n < \infty, n \geq 1$. In this case for every $n \geq 1$ operator \mathcal{A}_n is an operator with discrete spectrum. But an inverse of the direct sum operator $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$ is not compact operator in $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, because $\dim \mathcal{H} = \infty$ and \mathcal{A} is an identity operator in \mathcal{H} .

Theorem 4.6. If $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n, \mathcal{A}_n$ is an operator with discrete spectrum in $\mathcal{H}_n, n \geq 1, \bigcap_{n=1}^{\infty} \rho(\mathcal{A}_n) \neq \emptyset$ and $\lim_{n \rightarrow \infty} \|R_{\lambda}(\mathcal{A}_n)\| = 0$, then \mathcal{A} is an operator with discrete spectrum in \mathcal{H} .

Proof. For each $\lambda \in \bigcap_{n=1}^{\infty} \rho(\mathcal{A}_n)$ we have $R_{\lambda}(\mathcal{A}_n) \in C_{\infty}(\mathcal{H}_n), n \geq 1$.

Now define the following operators $\mathcal{K}_m : \mathcal{H} \rightarrow \mathcal{H}, m \geq 1$ as

$$\mathcal{K}_m := \{R_{\lambda}(\mathcal{A}_1)u_1, R_{\lambda}(\mathcal{A}_2)u_2, \dots, R_{\lambda}(\mathcal{A}_m)u_m, 0, 0, \dots\}, u = (u_n) \in \mathcal{H}.$$

The convergence of the operators \mathcal{K}_m to the operator \mathcal{K} in operator norm will be investigated.

For the $u = (u_n) \in \mathcal{H}$ we have

$$\begin{aligned} \|\mathcal{K}_m u - \mathcal{K}u\|_{\mathcal{H}}^2 &= \sum_{n=m+1}^{\infty} \|R_{\lambda}(\mathcal{A}_n)u_n\|_{\mathcal{H}_n}^2 \leq \sum_{n=m+1}^{\infty} \|R_{\lambda}(\mathcal{A}_n)\|^2 \|u_n\|_{\mathcal{H}_n}^2 \\ &\leq \left(\sup_{n \geq m+1} \|R_{\lambda}(\mathcal{A}_n)\| \right)^2 \sum_{n=1}^{\infty} \|u_n\|_{\mathcal{H}_n}^2 = \left(\sup_{n \geq m+1} \|R_{\lambda}(\mathcal{A}_n)\| \right)^2 \|u\|_{\mathcal{H}}^2 \end{aligned}$$

thus we get $\|\mathcal{K}_m u - \mathcal{K}u\| \leq \sup_{n \geq m+1} \|R_{\lambda}(\mathcal{A}_n)\|, m \geq 1$.

This means that sequence of operators (\mathcal{K}_m) converges in operator norm to the operator \mathcal{K} . Then by the important theorem of the theory of compact operators it implies that $\mathcal{K} \in C_{\infty}(\mathcal{H})$ [1], because for any $m \geq 1$, $\mathcal{K}_m \in C_{\infty}(\mathcal{H})$.

Finally, using the Theorem 4.6 can be proved the following result.

Theorem 4.7. If $A_n^{-1} \in C_{\infty}(\mathcal{H}_n), n \geq 1$, $\sup_{n \geq 1} (b_n - a_n) < \infty$ and the sequence of first minimal eigenvalues $\lambda_1(\mathcal{A}_n)$ of the operators $\mathcal{A}_n, n \geq 1$ is satisfy the condition

$$\lambda_1(\mathcal{A}_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

then the extension $\tilde{L} = \bigoplus_{n=1}^{\infty} L_n$ is an operator with discrete spectrum in L^2 .

Theorem 4.8. Let $H = \bigoplus_{n=1}^{\infty} H_n, A = \bigoplus_{n=1}^{\infty} A_n$ and $A_n \in C_p(H_n), n \geq 1, 1 \leq p < \infty$. In order for $A \in C_p(H)$ the necessary and sufficient condition is that the series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^p(A_n)$ be convergent.

Now we will dedicate an application the last theorem.

For all $n \geq 1, \mathfrak{H}_n$ is a Hilbert space, $\Delta_n = (a_n, b_n), -\infty < a_n < b_n < a_{n+1} < \dots < \infty, A_n : D(A_n) \subset \mathfrak{H}_n \rightarrow \mathfrak{H}_n, A_n = A_n^* \geq E, W_n : \mathfrak{H}_n \rightarrow \mathfrak{H}_n$ is unitary operator, $A_n^{-1}W_n = W_n A_n^{-1}, L_{W_n} u_n = u'_n + A_n u_n, A_n W_n^1(\mathfrak{H}_n, \Delta_n) \subset W_2^1(\mathfrak{H}_n, \Delta_n), H_n = L^2(\mathfrak{H}_n, \Delta_n), D(L_{W_n}) = \{u_n \in W_2^1(\mathfrak{H}_n, \Delta_n) : u_n(b_n) = W_n u_n(a_n)\}, L_{W_n} : H_n \rightarrow H_n, W = \bigoplus_{n=1}^{\infty} W_n, L_W = \bigoplus_{n=1}^{\infty} L_{W_n}, H = \bigoplus_{n=1}^{\infty} H_n$ and $h = \sup_{n \geq 1} (b_n - a_n) < \infty$.

Since for all $n \geq 1$ W_n is a unitary operator in \mathfrak{H}_n , then L_{W_n} is normal operator in H_n [10]. Also for $L_W : D(L_W) \subset H \rightarrow H$, the relation $L_W L_W^* = L_W^* L_W$ is true, i.e. L_W is a normal operator in H . It is known that, if $A_n^{-1} \in C_p(\mathfrak{H}_n)$, for $p > 1$, then $L_{W_n}^{-1} \in C_p(H_n), p > 1$ for all $n \geq 1$ [10]. On the other hand, if $A_n^{-1} \in C_{\infty}(\mathfrak{H}_n), n \geq 1$, then eigenvalues $\lambda_q(L_{W_n}), q \geq 1$ of operator L_{W_n} is in form

$$\lambda_q(L_{W_n}) = \lambda_m(A_n) + \frac{i}{a_n - b_n} \left(\arg \lambda_m(W_n^* e^{(-A_n(b_n - a_n))}) + 2k\pi \right), m \geq 1, k \in \mathbb{Z}, n \geq 1,$$

where $q = q(m, k) \in \mathbb{N}, m \geq 1, k \in \mathbb{Z}$. Therefore we have the following corollary.

Theorem 4.9. If $A = \bigoplus_{n=1}^{\infty} A_n$, $\mathfrak{H} = \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$ and $A^{-1} \in C_{p/2}(\mathfrak{H})$, $2 < p < \infty$, then $L_W^{-1} \in C_p(H)$.

Proof. The operator L_W is a normal in H . Consequently, for the characteristic numbers of normal operator L_W^{-1} an equality $\mu_q(L_W^{-1}) = |\lambda_q(L_W^{-1})|$, $q \geq 1$ holds [1]. Now we search for convergence of the series $\sum_{q=1}^{\infty} \mu_q^p(L_W^{-1})$, $2 < p < \infty$.

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \mu_q^p(L_W^{-1}) &= \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{m=1}^{\infty} \left(\lambda_m^2(A_n) + \frac{1}{(b_n - a_n)^2} (\delta(m, n) + 2k\pi)^2 \right)^{-p/2} \\ &\leq \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{m=1}^{\infty} \left(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n - a_n)^2} \right)^{-p/2} \\ &\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\lambda_m^2(A_n))^{-p/2} + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n - a_n)^2} \right)^{-p/2} \end{aligned}$$

where $\delta(m, n) = \arg \lambda_m(W_n^* e^{(-A_n(b_n - a_n))})$, $n \geq 1, m \geq 1$. Then from the inequality $\frac{|ts|}{t^2 + s^2} \leq \frac{1}{2}$ for all $t, s \in \mathbb{R} \setminus \{0\}$ and last equation we have the inequality

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n - a_n)^2} \right)^{-p/2} \leq 2^{-p} \pi^{-p/2} h^{p/2} \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \frac{1}{\lambda_m(A_n)} \right|^{p/2} \sum_{k=1}^{\infty} \left| \frac{1}{k} \right|^{p/2} \right)$$

Since $A^{-1} \in C_{p/2}(\mathfrak{H})$, then the series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p/2}$ is convergent. Thus the series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n - a_n)^2} \right)^{-p/2}$$

is also convergent. Then from the relation

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p/2}$$

and the convergence of the series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p/2}$ we get that the series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p}$ is convergent too. Consequently the series $\sum_{q=1}^{\infty} \mu_q^p(L_W^{-1})$, $2 < p < \infty$ is convergent and thus $L_W^{-1} \in C_p(H)$, $2 < p < \infty$.

The Theorem 4.8 and 4.9. can be generalized.

Corollary 4.10. Let for all $n \geq 1$ $A_n \in C_{p_n}(H_n)$, $1 \leq p_n < \infty$ and $p = \sup_{n \geq 1} p_n < \infty$. For

$A \in C_p(H)$ the necessary and sufficient condition is that the series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^p(A_n)$ be convergent.

Theorem 4.11. If $A_n^{-1} \in C_{p_n/2}(H_n)$, $2 \leq p_n < \infty$, $p = \sup_{n \geq 1} p_n < \infty$, then $L_W^{-1} \in C_p(H)$.

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